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# QUASI-MONTE CARLO ALGORITHM FOR PRICING OPTIONS

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**Abstract**

*The purpose of this paper is to compare the use of Quasi-Monte Carlo methods, especially the use of recent developed (t, m, s)-nets, versus classical Monte Carlo method for valuing financial derivatives. Some research has indicate that under certain condition Quasi-Monte Carlo is superior than the traditional Monte Carlo in terms of rate of convergence and accuracy. In particular, theoretic results hinted that the so-called (t, m, s)-net suppose to be the most powerful one among all the Quasi-Monte Carlo methods when the problem is "smooth". However, the application of (t, m, s)-net was not included in the existing simulation literatures. In this paper I will introduce the algorithms of generate the most common Quasi-Monte Carlo sequences, then implement these sequences in several path-dependent options. Our investigation showed that Quasi-Monte Carlo methods outperform the traditional Monte Carlo.*

## I. Introduction

Monte Carlo method is widely used in pricing financial derivatives and measuring their risks. One primary reason for this phenomenon is it is easy to apply and this method for numerical integration gives errors, whose order of magnitude, in terms of the number of nodes, is independent of the dimension. However the stochastic nature of the Monte Carlo method causes some unpleasant drawbacks. For example, it is difficult, if not impossible, to generate a sequence of high quality

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random samples, and also the rate of convergence is only proportional to $N^{-1/2}$ which is very slow. This motivates the search for methods which converge faster.

Some research has been done in this area and interesting results have been obtained. For example, Paskov (1996) developed software to generate Sobol and Halton quasi-Monte Carlo sequences and compared their performances with the Monte Carlo method for mortgage-backed securities. Papageorgiou and Traub (1996) analyzed various techniques for valuing a typical collaterized mortgage obligation where the problem was reduced to numerically evaluating an integral over a 360-dimensional unit cube and concluded that quasi-Monte Carlo methods converged significantly faster than Monte Carlo methods and attained small errors, even while using a small number of points. Acworth, Broadie, and Glasserman (1996) made a detailed comparison of some Monte Carlo and quasi-Monte Carlo techniques for the pricing of moderate and high dimensional options. They found that quasi-Monte Carlo methods outperform ordinary Monte Carlo methods. In this paper we show that for three different types of options, the use of another quasi-Monte Carlo technique, namely $(t, m, s)$-nets, also outperforms Monte Carlo methods.

The rest of the paper is arranged as follows: in Section II, we give some basic definitions which are related to derivatives pricing, quasi-Monte Carlo and $(t, m, s)$-net. In Section III, we introduce the exotic options to be valued. In Section IV we discuss methods of constructing $(t, m, s)$-nets and other Quasi Monte Carlo sequences. Finally in Section V, we shall give some numerical comparisons results between the use of Monte Carlo methods and $(t, m, s)$- nets in pricing the mentioned options.

## II. Monte Carlo and Quasi-Monte Carlo for Financial Derivative Pricing

The present value of any derivative security is the discounted value of its expected terminal date cash flow:

$$\text{Price} = e^{-rT}\, E[f(S_0, \ldots, S_T)]$$

where $T$ is the maturity date of the derivative, $E[\cdot]$ is the expectations operator under the risk-neutral measure, $f$ is the derivative's terminal date cash flow, which may depend on the entire price history of the underlying asset, and $S_0, \ldots, S_T$ is the history of prices for the underlying assets from $t = 0$ to $T$. Here the expectation can be represented as an integration over a very high dimensional domain. Therefore pricing any derivative security can be interpreted as evaluation a high dimensional integral.

Monte Carlo and Quasi-Monte Carlo methods are the most commonly used general method for high dimensional integral evaluations.

The idea underlying the Monte Carlo method is to replace the integral of $f(x)$, which is a continuous average, by a discrete average over randomly chosen points. More precisely, we have the following approximation:



$$\int_{[0,1]^s} f(x)dx \approx \frac{1}{N}\sum_{n=1}^{N} f(x_n) \qquad (1)$$

where $x_1, x_2, \ldots, x_N$ are random points which uniformly distribute on $[0, 1]^s$. The key issue is how to choose these points so that the error

$$\left| \int_{[0,1]^s} f(x)dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right|$$

is as small as possible. The Monte Carlo method used random numbers and by large number theory the expected error is $O(N^{-1/2})$ for $N$ sample paths.

One class of modified Monte Carlo methods are often known as quasi-Monte Carlo (or low-discrepancy) algorithms. A quasi-Monte Carlo method also approximates an integral by a discrete average, except that the random samples in the Monte Carlo method are replaced by well-chosen deterministic points, the low discrepancy points. A quasi-Monte Carlo simulation can provide a much improved convergence rate, close to $O(N^{-1})$ or even $O(N^{-3/2})$ in some special cases. This improvement in convergence rate has the potential for significant gains both in computational time and in the range of applications of simulation methods for finance problems. In another word, a quasi-Monte Carlo method can be described as a deterministic version of a Monte Carlo method in the sense that the random samples in the Monte Carlo method are replaced by deterministic points so a deterministic error bound can be established. Since all points are chosen explicitly rely on number theory idea, the quasi-Monte Carlo methods are often called number theoretic methods.

Quasi-Monte Carlo or low discrepancy methods have received substantial attention in the financial literature recently. Let us now present some fundamental concepts which related to a quasi-Monte Carlo method or low discrepancy sequence. Discrepancy is a measure of deviation from uniformity of a sequence of real numbers. In particular, the star-discrepancy of $N$ points $x_1, x_2, \ldots, x_N \in [0, 1]^s$, is defined by

$$D(N)^* = \sup_J |A(J;N) \quad V(J)N|.$$

Here A(J; N) is the number of $n$, $1 \le n \le N$, with $x_n \in J$ and $V(J)$ is the volume of the subinterval $J$ and the supremum is extended over all subintervals $J$ of the form $J = \Pi_{i=1}^{s}[0, u_i]$.

A set of $x_1, x_2, \ldots, x_N \in [0, 1]^s$ is a low discrepancy if $D_N^*(P)$reaches its lowest asymptotic range, namely $O(N^{-1} (\log N)^s)$ for all $N > 1$.

The following Koksma-Hlawka inequality establishes the relationship between low discrepancy sequences and integration (see page 19 in Niederreiter, 1992):

$$\left|\int_{[0,1]^s} f(x)dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \le V(f)D_N^*(x_1,\ldots,x_N), \tag{2}$$

which holds for any function $f$ of s variables that has bounded variation $V(f)$ on $[0, 1)^s$ and for any $x_1, \ldots, x_N \in [0, 1)^s$. Some known low discrepancy point sets include Sobol, Halton, Faure and $(t, m, s)$-net. Roughly speaking, these points sets all have the property that the rate of convergence is at least proportional to $N^{-1}(\log N)^s$. The $N^{-1}$ factor in the convergence formula for low discrepancy points may be contrasted with the $N^{1/2}$ convergence of Monte Carlo and suggest that low discrepancy methods are superior to Monte Carlo methods (at least, in theory, for not so large $s$).

## III. Examples of Options

In this section we give examples of options whose payoffs allow closed-form pricing or a numerical procedure so that one can obtain a precise value for the option. We shall use three options to test our algorithm and report our results in Section V.

*Discrete down-and-out call.* This is a barrier option on a single underlying asset with payoff

$$C_B = max(S_T - K, 0)1_{\{min_{1\le i\le m} S_{t_i} > H\}}$$

and price $e^{-rT} E|C_B|$. Here $K$ and $H$ are constants and $t_1, \ldots, t_m$ are points in $[0, T]$. This is a standard call option struck at $K$ unless the underlying asset is below the barrier $H$ at any of the monitoring dates $t_1, \ldots, t_m$, at which the option is knocked out and the holder receives nothing. This option can be priced in closed-form (see Hull (1997) for details).

*Discrete average rate option.* This is perhaps the simplest and most frequently encountered option for which Monte Carlo methods are used as a primary pricing for its payoff, which can be described as follows:

$$C_A + max\left(0, \frac{1}{m}\sum_{i=1}^{m} S_{t_i} - K\right)$$

This is an option on the average price of the underlying asset over a fixed set of monitoring dates. Its intractability arises from the fact that the sum of lognormal random variables does not, in general, admit a closed-form distribution. However we can get a close form solution of its surrogate check (page 466, Geman and Marc Yor (1993) for detail).

*A Multi-asset option.* This example has the following payoff



$$C_m = max\left(0, (\prod_{i=1}^{m} S_T^i)^{\frac{1}{m}} - K\right),$$

where $S_T^1, \ldots, S_T^m$ are the terminal prices of m correlated assets, the geometric average appearing in the payoff leads to a closed-form solution against which to compare simulation results. Minor variations in the payoff function, however, lead to pricing problems with no closed-form solution.

## IV. $(t, s)$-sequence and $(t, m, s)$-nets

We give the methods for constructing low discrepancy point sets, especially $(t, m, s)$-net. One class of low discrepancy sequence such as Sobol, Halton, Faure are usually called $(t, s)$-sequence, and another class is $(t, m, s)$-nets. According to Niederreiter (1992), $(t, m, s)$-nets yields the smallest discrepancy bound among all the low discrepancy sets and hence, by the Koksma-Hlawka inequality, the smallest error bound (within the class of functions of bounded variation in the sense of Hardy and Krause) among all known constructions of point sets. Especially, within the class of functions with rapidly converging Walsh series, Larcher and Traunfellner (1994) have shown that digital $(t, m, s)$-nets yield an error bound of the optimal order of magnitude. Therefore, by the Koksma-Hlawka inequality, $(t, m, s)$-net should be more efficient for high dimensional integral evaluation when it is used with the quasi-Monte Carlo method.

Neiderreiter (1992) gives a general method for constructing $(t, s)$-sequences; here we give a example of constructing Faure sequence. For a prime number $b \ge s$ and $N = 0, 1, 2, \ldots$ consider the base $b$ representation of $N$, i.e., $N = \sum_{i=0}^{\infty} a_i(N)b^i$, where $a_i(N) \in [0, b]$ are integers, $i = 0, 1, \ldots$ The *j-th* coordinated of the points $x_N$ is then given by

$$x_N^{(j)} = \sum_{i=0}^{\infty} x_{Nk}^{(j)} b^{-k-1}, 1 \le j \le s,$$

where $x_{nk}^{(j)} b^{-k-1} = \sum_{i=0}^{\infty} c_{ks}^{(j)} a_s(N)$. The matrix $C^{(j)} = (c_{ks}^{(j)}$ is given by $C^{(j)} = A^{(j)} P^{j-1}$, where $A^{(J)}$ is a nonsingular lower triangular matrix and $P^{j-1}$ is the $j$–1 power of the Pascal matrix.

Our main interest is to introduce the latest developed low discrepancy points, the $(t, m, s)$-net. There are many methods to construct $(t, m, s)$-nets (see Clayman, et al. (to appear)). The most commonly used methods include: direct constructions using various properties of finite fields and polynomials over finite fields, error-correcting codes including both linear and nonlinear codes such as Kerdock codes, combinatorial methods including generalized orthogonal arrays, and a method which uses linear combinations of the rows of a so-called *generator matrix* (see

Bierbrauer and Edel, 1998). We will focus our attention on this last method as it is particularly suitable for parallel machine implementation. The purpose of our current paper is the parallel computation, testing and analysis of nets for use in high dimensional numerical integration problems that arise in finance, our purpose is not to discuss the theory and techniques for the actual construction of generator matrices which are described in Bierbrauer and Edel (1998). We thus only briefly sketch the use of generator matrices in the construction of nets. The idea of constructing a $(t, m, s)$-net using generator matrices is based on ideas developed by Bierbrauer and Edel (1998), and described in their work. Here we give a parallel version of the method. A $(t, m, s)$-net is obtained as follows. For a prime number say $b = 2$. Consider a $(t, m, s)$-net in base $b = 2$, find a generator matrix from Bierbrauer and Edel (while the generator matrices are all binary and thus have $b = 2$, a similar method could be used for nets in any prime power base $b$ by constructing generator matrices over the finite field $F_b$). The generator matrix given in Bierbrauer and Edel (1998) is a matrix of $m$ rows, and $s$ blocks, where each block consists of $m - t - 1$ column vectors each of length $m$. Assume we are working with $p$ processors. Given $m$, we first need to distribute the set of points $N_m = \{0, 1, 2, \ldots, 2^m - 1\}$ evenly into all $p$ processors. We need to decompose this set into roughly $p$ subsets with each subset containing roughly $n_p \equiv \left[\frac{2^m}{p}\right]$ elements (here $[\xi]$ represent the largest integer less than $\xi$). One naive strategy is the following decomposition by cutting the original set into segments in a natural order as follows: First let each process to take one of the first $p$ numbers in the set, namely $\{0, 1, 2, \ldots, p-1\}$, and then repeat this process to the subset $\{p, p+1, \ldots, 2p-1\}$ and continue until all the numbers are taken. For each $i \in N_m$, the processor that contains $i$ calculates the base 2 expansion of $i$ which is in the form $i = a_0 + a_1 2 + \ldots + a_{m-1} 2^{m-1}$ with $a_i = 0$ or $a_i = 1$. We then form the modulo 2 linear combination of the rows of the extended binary generator matrix corresponding to the base 2 expansion of $i$. The coefficients in the resulting linear combination are actually the digits in the binary expansion of $i$. Then we split the result into s groups from left to right, each group having $m - t$ digits. We convert the values from each group from binary to the corresponding base 10 number and then multiply the resulting real vector (which has s entries) by $2^{-s}$. This gives the $i$-th net point $x_i$.

For detail and example of this construction we refer to Li and Mullen (2000).

## V. Numerical Comparisons

In this section we compare the performance of the Monte Carlo method with that of $(t, m, s)$-nets in the valuation of three classes of options: the discrete average option, the multi-asset option in which all underlying assets are independent and the multi-asset option in which all underlying assets have correlation 0.3.

For the standard Monte Carlo method we use the random number generator RAN1 from the Numerical Recipes in C. For the $(t, m, s)$-net method we will use



the method of generator matrices from Bierbrauer and Edel (1998) as discussed in Section IV. For the Faure sequence we generated by the mentioned method.

We perform our tests on problems with dimension $s = 10$ and we use $N = 2^{14}$ points. We use a (5, 14, 10)-net in base 2. In this example we have $t = 5$, $m = 14$ and $s = 10$ and so the net contains $2^{14} = 16{,}384$ points. The generator matrix for our net was obtained from Bierbrauer and Edel (1998). We now compare the performance of the ordinary Monte Carlo method using $2^{14}$ points with that of a (5, 14, 10)-net and Faure also using $2^{14}$ points. We can easily observe from the following pictures the errors between the exact solution and Monte Carlo, also the errors between the exact solution and (5, 14, 10)-net, the error between the exact solution and Faure. The (5, 14, 10)-net and Faure obviously outperform than Monte Carlo. From left to right these small pictures show the numerical comparison on the discrete average option, the multiasset option in which all underlying assets are independent and multi-asset option in which all underlying assets have correlation 0.3.

FIGURE 1

ERRORS BETWEEN THE EXACT SOLUTION AND MC (DASHED LINE) AND NET (SOLID LINE)

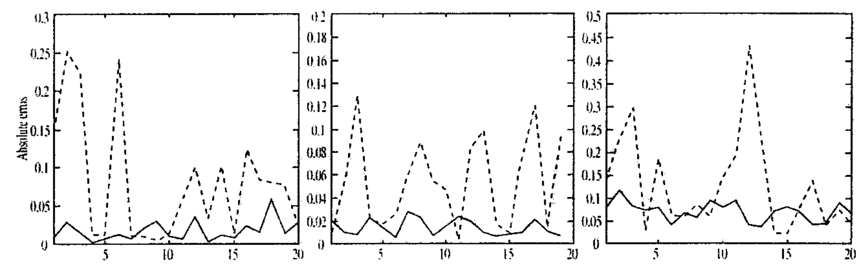
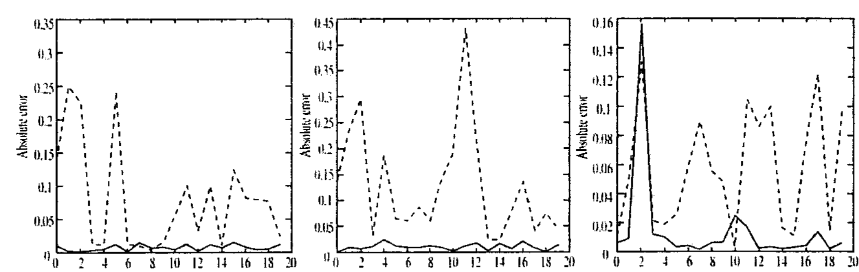



FIGURE 2

ERRORS BETWEEN THE EXACT SOLUTION AND MC (DASHED LINE) AND FAURE (SOLID LINE)



According to the tables from Clayman et al. (1999), the (5, 14, 10)-net has the smallest known value of t for the given values $m = 14$ and $s = 10$. It is however not known whether a $(t, 14, 10)$-net in base 2 can exist with $t = 3$ or $t = 4$, although it is known that t cannot be less than 3 in order for such a net to exist. We remind the reader that for fixed m, the smaller the value of t the more uniform is the distribution of points in $[0, 1)^{10}$ and thus it's possible that the estimates given above could indeed be improved if a net with a smaller value of t could be constructed.

## VI. Conclusion

The main focus of this paper is to introduce the use of $(t, m, s)$-nets in valuing derivatives. From the examples we have tested, the discrete average rate option, the multi asset option with no correlation, and the multi asset option with 0.3 correlation, it appears that Faure and $(t, m, s)$-nets outperform the ordinary Monte Carlo method. But the comparison between Faure and $(t, m, s)$-net is inconclusive so far. So further investigation is of course required, for example, more tests need to be run for higher dimensions s for different nets. Also these methods need to be applied to other types of derivatives and securities. Such testing is now underway.

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