Efficiency of Two Sided Investments in an Equilibrium Unemployment Framework*

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Abstract

This paper investigates the efficiency of investments by firms and workers in a matching model with high- and low-productivity jobs. Search is sector specific and random within sectors. Search frictions and ex-post bargaining imply that wage inequality arises as a result of the difference in investment costs between the sectors. The efficiency properties of the equilibrium are analyzed under the particular division in bargaining proposed by Hosios (1990). The conclusion is that the equilibrium is inefficient, with a too low fraction of workers and a too high vacancy-unemployment ratio in the high-productivity sector. The opposite happens in the low-productivity sector.

1 Introduction

This paper analyzes the efficiency of two sided investments in a labor market model with search frictions. It presents a model where firms and workers make ex-ante investments, search for suitable partners and once they match they bargain over the surplus that results from their investments.

Typically, ex-ante investments and ex-post bargaining imply under-investments. When two parties commit to invest in advance to make production possible, investments are sunk at the bargaining stage and then the parties do not get the full marginal benefits of their investments. It is essential for this result that rents arise because of the presence of asset specificity. In the extreme case, investments are only productive within the relationship and the outside options are zero.

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In a labor market there are a lot of agents on each side of the market taking investments decisions simultaneously and intuitively the degree of asset specificity is significantly reduced. Indeed, Cole, Mailath and Postlewaite (2001a, 2001b) and also Felli and Roberts (2002) analyze matching models where competition for partners can prevent holdup problems. From this perspective, under-investment problems are mitigated in a frictionless labor market. However, in a labor market with search frictions rents from matching take place and they bring back some degree of asset specificity to the productive relationship creating again incentives to under-investments (Acemoglu and Shimer, 1999).

This paper analyzes the efficiency properties of a standard two-sector search and matching model with two-sided investments. There exist a high- and a low-productivity sector where firms and workers participate as a result of their ex-ante allocation decisions. The model generates a sectorial distribution of firms and workers. Heterogeneity arises because workers have to pay for a cost to become high skilled and firms, correspondingly, need to make higher investments in order to open a vacancy in the high-productivity sector. It is assumed that a high-productivity vacancy can only be filled by a high-skill worker and low-skill workers can only be productive in the low-productivity sector. Also, any filled vacancy employs one worker to produce one unit of a good. The number of vacancies in each sector is determined by two free-entry conditions which drive expected profits from job creation to zero. The distribution of the workforce is the one that equalizes the expected net income of high- and low-skill workers. The model is related to the two sector models of Acemoglu (2001) and Albrecht and Vroman (2002) with the main difference being that the skill composition of the workforce is endogenous.

The general conclusion of previous research studying two-sided investment in matching models is that with search frictions both workers and firms under-invest in human and physical capital, respectively. This is not necessarily what happens in the model presented here. In fact, it is shown that under the particular division in bargaining satisfying the Hosios condition with costly investments workers under-invest and firms tend to over-invest in the high-productivity sector. This result is explained by the interaction of investment externalities with their general equilibrium effects. Of particular importance is the endogenous price assumption introduced to the model.

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1 It is not considered the possibility of mismatch between high-skill workers and low-productivity firms previously analyzed in the literature (McKenna, 1996; Gautier, 2000; Albrecht and Vroman, 2002; Uren 2006; Dolado, Jansen and Jimeno, 2009). Notwithstanding, the conclusions of the paper are robust to this potential extension for high enough investment costs in the high-productivity sector. See Blázquez and Jansen (2008) for an efficiency analysis of Albrecht and Vroman (2002).

2 Aricò (2009) presents a model with endogenous types in both sides of the market and analyzes the effect of subsidies to education and innovation on output and unemployment. Apart from not analyzing the efficiency properties of his model, the treatment of frictions is quite different from this paper.

Each sector requires capital and labor to produce an intermediate good that is sold in a perfectly competitive market and used with the other sector’s output to produce a final consumption good (Acemoglu, 2001). As prices depend on the relative supply of the intermediate goods, and therefore the number of agents in each sector, a higher investment cost of any good reduces its supply and increases its price, which is the value of output in each productive relationship (i.e. the match surplus). Given that wages are determined by Nash Bargaining over the match surplus, the sectorial price difference resulting from different investment costs explains wage inequality in the model. These price effects have important implications for the efficiency properties of the equilibrium. Even when search frictions and rent sharing give incentives to under-investments by firms and workers, the relative scarcity of goods produced by the high-productivity sector implies that the sectorial price difference is higher than it should be. This incentive to invest in the high-productivity sector mitigates under-investment problems. With endogenous prices there is also more interdependence in the investments decisions in the economy because they do not only rely on investment costs and matching probabilities but also on their effect on prices.\(^4\)

It is assumed that search is sector specific but random within sectors. That is, given the matching requirements, agents only search for the partners with whom production is possible. This leads to a labor market with two separate matching functions, one for each sector. Equilibrium in standard matching models involves trading externalities: a firm posting a vacancy makes it more difficult for the other vacancies to meet an unemployed worker (congestion externality) and easier for the unemployed workers to meet an unfilled vacancy (thick market externality). These externalities cancel out only when the measure of workers’ bargaining power $\beta$ is equal to the elasticity of the matching function with respect to unemployment (Hosios, 1990; Pissarides, 2000). The efficiency analysis of this paper is done assuming the Hosios condition holds. This makes it possible to identify the efficiency effects of investments not related to the standard ones in this type of models. As previously shown by Acemoglu and Shimer (1999), under the sector specific matching assumption the Hosios condition makes firms internalize the externalities associated to vacancy creation. An interesting result of the paper is that there is an externality in the determination of the skill composition that is not related to the matching assumption. This arises because when workers take their skill decision they do not consider how a better skill composition increases the average value of matches in the economy. These investment externalities together with the general equilibrium effects of the model (in particular the price effects) determine the final efficiency properties of the model.

The paper is structured as follows: The next section describes the general environment. Section

\(^4\) Additionally, in line with some empirical literature (Autor, Katz and Kearney, 2005 and 2008) the model with endogenous prices is consistent with a net supply of skills explanation of wage inequality.
3 presents the equilibrium and Section 4 analyzes the efficiency properties of the equilibrium. Section 5 concludes.

2 Model

2.1 Basic Assumptions

Consider a continuous-time economy populated by a unit mass of workers producing a final consumption good $Y$ with two intermediate inputs $Y_b$ and $Y_g$ according to a CES technology,

$$ Y = (\alpha Y_b^\rho + (1 - \alpha)Y_g^\rho)^{1/\rho}. \quad (1) $$

Following Acemoglu (2001), $b$ and $g$ refer to a bad-job and a good-job sector, respectively. Good $Y$ is sold in a competitive market at a normalized price of 1. It is assumed that $\rho < 1$ and $\alpha$ measures the share of $Y_b$ in the final good production function. The first order conditions of the aggregate economy problem lead to the following equilibrium prices for the two intermediate goods,

$$ p_b = \alpha Y_b^{\rho-1}Y^{1-\rho} \quad (2) $$
$$ p_g = (1 - \alpha)Y_g^{\rho-1}Y^{1-\rho}. \quad (3) $$

These two intermediate goods are produced via a Leontief technology using capital and labor. That is, production of one unit of an intermediate good $i = b, g$ takes place when a firm with a vacancy meets a suitable unemployed worker. Then, $Y_i$ measures not only aggregate production but also aggregate employment in sector $i$.

Firms can only produce in one of the two sectors and need to make ex-ante investments before production takes place. Denote by $k_i$ the firm investment cost in sector $i$ and assume that investments are generally more costly in the $g$ sector, that is $k_g \geq k_b$. Similarly, workers can only participate in one of the two sectors. Workers producing in sector $g$ must have acquired the necessary skills at a cost $c$. It is assumed that having incurred the cost $c$ precludes workers from being able to produce in sector $b$.\(^5\) As a result of the allocation of workers across the sectors there is a fraction $\pi$ of workers who can only work in sector $b$. As long as it is more costly for either firms or workers to participate in sector $g$ than in sector $b$, $Y_b > Y_g$ and therefore $p_g > p_b$.

\(^5\)We can relax this assumption to allow for high-skill workers to be productive in both sectors. However, even under a more general assumption about the skill requirements of the different jobs, it can be shown that for a high enough price difference between the sectors high-skill workers would never find it worthwhile to accept a $b$ sector offer.
2.2 Search and Meeting Process

At any moment in time, a job is either filled or vacant and a worker is either employed or unemployed. Unemployed workers and firms offering vacancies have to spend resources in order to meet each other. When a vacant job is filled with a qualifying unemployed worker, production takes place and the job generates a rent. As standard in this literature workers and firms meet according to a matching function \( M(u, v) \) where \( u \) is unemployment and \( v \) vacancies. This matching function is twice differentiable, increasing in both arguments and has constant returns to scale.

Given that different jobs have different matching requirements firms and workers could only search for their potential partners with whom production is possible. This leads to the sector specific search assumption adopted in this section. There are then two matching functions, one for bad jobs and another for good jobs.

The constant returns to scale assumption implies that \( M(u_i, v_i)/v_i = q(\theta_i) \), where \( \theta_i = v_i/u_i \) is defined as the tightness of the labor market in sector \( i \). The function \( q(\theta_i) \) represents the flow rate at which vacancies meet unemployed workers and is decreasing in \( \theta_i \). In addition, \( M(u_i, v_i)/u_i = \theta_i q(\theta_i) \) is the flow rate at which unemployed workers meet unfilled vacancies and is increasing in \( \theta_i \).

In steady state, unemployment persists because some of the existing jobs break up at the exogenous rate \( s \) providing a flow into unemployment. When a job breaks up, the worker becomes unemployed and the job becomes an unfilled vacancy.

Given the intermediate production technology, intermediate good production and sectorial employment are equivalent,

\[
Y_b = \pi - u_b \quad (4) \\
Y_g = 1 - \pi - u_g. \quad (5)
\]

The differential costs of operating in either the bad or the good-job sector imply production and price differences between the sectors. That is, since producing in sector \( g \) is more costly than producing in sector \( b \) prices are relatively higher in sector \( g \) and also \( Y_b > Y_g \).

2.3 Bellman Equations and Wage Determination

Denote by \( J_i^U \) the value of unemployment, \( J_i^E \) the value of employment, \( J_i^V \) the value of an unfilled vacancy and \( J_i^F \) the value of a filled job in sector \( i \). These values are linked by the following flow equations:

\[
r J_i^U = \theta_i q(\theta_i) (J_i^E - J_i^U). \quad (6)
\]
Equation (6) states that the flow value of unemployment in any sector is equivalent to the expected capital gain from finding a job and realizing a flow value \( rJ_i^E \). In a similar way the flow value of employment in sector \( i \) is
\[
 rJ_i^E = w_i + s(J_i^U - J_i^E); \tag{7}
\]
that is, the flow value for an employed worker in sector \( i \) equals the wage \( w_i \) plus the expected capital loss \( s(J_i^U - J_i^E) \).

Similarly, the flow values for a firm of having a filled job and a vacancy are
\[
 rJ_i^F = p_i - w_i + s(J_i^V - J_i^F) \tag{8} \\
 rJ_i^V = q(\theta_i)(J_i^F - J_i^V), \tag{9}
\]
respectively.

**Generalized Nash Bargaining.** As standard in this literature, when a match is formed a surplus equivalent to \( p_i \) is realized and the wage is determined by rent sharing. There is a bilateral negotiation between the parties; a wage is a solution to a generalized Nash Bargaining problem with threat points equal to the worker’s and the firm’s respective continuation values. Then,
\[
(1 - \beta)(J_i^E - J_i^U) = \beta(J_i^F - J_i^V) \tag{10}
\]
where \( \beta \) denotes the exogenous workers’ share of the surplus. The wage assumption implies that the rent implied by the production of a \( g \) sector good is transferred in part to the employees via higher wages.

### 2.4 Equilibrium conditions

1. **Free entry condition.** The number of vacancies in any sector is determined by a free entry condition which drives expected net profits for opening vacancies to zero,
\[
 rk_i = rJ_i^V. \tag{11}
\]

2. **Indifference condition.** The equilibrium distribution of workers is determined by a non arbitrage condition by which workers are indifferent between the two sectors. That is, in equilibrium all the workers have the same expected value of unemployment net of investment costs,
\[
 J_i^U - c = J_i^U. \tag{12}
\]

\footnote{It is assumed that the flow income from unemployment is zero. This is a convenient assumption that simplifies the calculations. Depending on parameterizations, results would extend though for positive unemployment income values.}
Any investment decision taken by any agent in this economy has two effects: Not only it affects the matching probabilities faced by other agents but also the surplus of the matches via the effect on prices. Then, an extra worker in sector $i$ increases the matching probability of $i$ vacancies and then leads to more production and lower prices in that sector.

3. Steady state unemployment. The flow out of unemployment from any sector has to be equal to the flow back into unemployment to that sector,

$$q(\theta_b)\theta_b u_b = s(\pi - u_b)$$  \hspace{1cm} (13)$$
$$q(\theta_g)\theta_g u_g = s(1 - \pi - u_g),$$  \hspace{1cm} (14)$$

and the steady state number of unemployed in $i$ is

$$u_b = \frac{s\pi}{q(\theta_b)\theta_b + s}$$  \hspace{1cm} (15)$$
$$u_g = \frac{s(1 - \pi)}{q(\theta_g)\theta_g + s}.$$  \hspace{1cm}

3 Equilibrium

In a steady state equilibrium firms and workers maximize their respective objective functions, given the matching and separation technologies and all the equilibrium conditions are satisfied. The equilibrium is defined as a tightness of the labor market $\theta_i$, workforce distribution, value functions and prices of both goods such that equations (2) to (15) are satisfied.

Since $Y_b = \pi - u_b$ and $Y_g = 1 - \pi - u_g$, the equilibrium intermediate goods prices are

$$p_b = \alpha \left( \frac{q(\theta_b)\theta_b\pi}{q(\theta_b)\theta_b + s} \right)^{\rho - 1} \left[ \alpha \left( \frac{q(\theta_b)\theta_b\pi}{q(\theta_b)\theta_b + s} \right)^\rho + (1 - \alpha) \left( \frac{q(\theta_g)\theta_g(1 - \pi)}{q(\theta_g)\theta_g + s} \right)^\rho \right]^{1 - \rho}/\rho$$  \hspace{1cm} (16)$$
$$p_g = (1 - \alpha) \left( \frac{q(\theta_g)\theta_g(1 - \pi)}{q(\theta_g)\theta_g + s} \right)^{\rho - 1} \left[ \alpha \left( \frac{q(\theta_b)\theta_b\pi}{q(\theta_b)\theta_b + s} \right)^\rho + (1 - \alpha) \left( \frac{q(\theta_g)\theta_g(1 - \pi)}{q(\theta_g)\theta_g + s} \right)^\rho \right]^{1 - \rho}/\rho.$$  \hspace{1cm}

Given that employment $Y_i$ increases with $\theta_i$ and the size of the workforce in $i$, we have that $p_i$ is decreasing in $\theta_i$ and increasing in $\theta_j$ (for $j = b, g$ and $j \neq i$). Also $p_g$ ($p_b$) is increasing (decreasing) in $\pi$.

Using equations (7) to (11), the wage equations are written in the standard way

$$w_i = \beta (p_i - rk_i) + (1 - \beta) r J_i^U.$$  \hspace{1cm} (17)$$

The wage in $i$ is a weighted average of the surplus that the firm gets (output per worker minus flow value of capital creation cost) and the flow value of unemployment.
The unemployment flow values are given by (6) with (7), (8), (9), and (17) substituted in,

\[ r J_i^U = \frac{\beta \theta_i q(\theta_i) \left( p_i - r k_i \right)}{r + s + \beta \theta_i q(\theta_i)}. \]  

(18)

The equilibrium flow values of vacancies are obtained from (8) and (9). After substitution, the equilibrium values of the key endogenous variables of the model are obtained from the next three equations:

\[ r J_b^V = r k_b = q(\theta_b) (1 - \beta) \frac{P_b}{D_b} \] 

(19)

\[ r J_g^V = r k_g = q(\theta_g) (1 - \beta) \frac{P_g}{D_g} \] 

(20)

\[ r c = q(\theta_g) \theta_g \beta \frac{P_g}{D_g} - q(\theta_b) \theta_b \beta \frac{P_b}{D_b} \] 

(21)

where

\[ D_i = r + s + (1 - \beta) q(\theta_i) + \beta \theta_i q(\theta_i). \]

Note that \( r J_i^V \) is decreasing in \( \theta_i \) as expected, and given the price effects, increasing in the other sector \( \theta_j \). Indeed, an increase in \( \theta_j \) is associated to a greater \( p_i \) and therefore \( r J_i^V \) (this result is derived formally in Appendix A). Given the properties of the matching function and the final output technology assumptions an equilibrium always exists. The argument is standard. Prices and the number of unemployed are uniquely determined by \( \theta_b \) and \( \theta_g \) and \( \pi \). The right hand side in equations (19) and (20) converges to infinity as \( \theta_i \rightarrow 0 \) and goes to zero as \( \theta_i \rightarrow \infty \). For any given \( \pi \), the steady state values of \( \theta_b \) and \( \theta_g \) are determined by the intersection of the loci formed by equations (19) and (20) with (16) substituted in. In Appendix A, it is shown that the loci are upward sloping and that they intersect only once like in Figure 1. Given the CES final output technology, for certain parameterizations it is also possible to obtain equilibria where only one sector prevails.\(^7\) It follows that, for every \( \pi \), there exists a unique equilibrium pair \((\theta_b, \theta_g)\). Moreover, given the price effects associated to changes in \( \pi \), both loci shift to the right with increases in \( \pi \). This means that the equilibrium \( \theta_g \) (\( \theta_b \)) increases (decreases) with \( \pi \) and therefore that \( J_g^U - J_b^U \) is monotonically increasing in \( \pi \). Additionally, given that \( J_g^U - J_b^U < 0 \) for \( \pi \rightarrow 0 \) and \( J_g^U - J_b^U > 0 \) as \( \pi \rightarrow 1 \), there is only one value of \( \pi \) (associated to a unique equilibrium pair \((\theta_b, \theta_g)\)) that satisfies (21).

In summary equations (19), (20) and (21) determine the unique triplet \((\theta_b, \theta_g, \pi)\) that solves the model. The solutions for \( u_b \) and \( u_g \) are obtained from the steady state equations (15).

\(^7\) For an analysis of the existence and uniqueness of the equilibrium in similar models see Cardullo (2009).
The next section analyzes how investments in the two sides of the market affect the efficiency properties of the equilibrium. For that reason it is useful to analyze the comparative statics of the equilibrium with respect to changes in the capital creation cost in the good-job sector \( (k_g) \) and the education cost \( (c) \). The results are summarized below,

**Remark 1** In equilibrium, \( \frac{\partial \theta_g}{\partial k_g} < 0, \frac{\partial \theta_b}{\partial k_g} < 0, \frac{\partial \pi}{\partial k_g} > 0, \frac{\partial \theta}{\partial c} > 0, \frac{\partial \theta_b}{\partial c} < 0, \frac{\partial \pi}{\partial c} > 0. \)

See Appendix C for a proof.\(^8\) In Figure 1, an increase in \( k_g \) shifts the good-job-locus equation to the left. The decrease in \( \theta_g \) implies a higher \( p_g \). The increase in relative prices in sector \( g \) is related to a decrease in \( \theta_b \) which restores the equilibrium along the bad-job-locus given by (19). Equations (19) and (20) then imply that \( \frac{\partial \theta_g}{\partial k_i} < 0 \) and \( \frac{\partial \theta_b}{\partial k_j} < 0 \). Given that the fall in \( \theta_g \) is greater than the fall in \( \theta_b \), the \( \pi \) satisfying in addition the non arbitrage condition (21) is greater. Intuitively, workers find it relatively less attractive to participate in sector \( g \) when the number of \( g \) vacancies falls.

Regarding the effect of the education cost \( c \), \( \frac{\partial \pi}{\partial c} > 0 \) results from (21). An increase in \( c \) requires a higher difference in the unemployment values between the sectors which happens via the increase

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\(^8\)It can also be shown that, as expected, \( \frac{\partial \theta_g}{\partial r} < 0, \frac{\partial \theta_b}{\partial s} < 0, \frac{\partial \pi}{\partial s} < 0. \)
in \( p_g - p_b \) resulting from a higher \( \pi \). The effect of \( c \) on labor market tightness is related to \( \frac{dc}{dc} > 0 \). All else equal, the greater number of low skilled workers resulting from an increase in \( c \) leads to a higher \( p_g \) and a lower \( p_b \). It follows that \( \frac{dp_g}{dc} > 0 \) and \( \frac{dp_b}{dc} < 0 \) to satisfy the respective free-entry conditions. Then \( \frac{dp_g}{dc} > 0 \) and \( \frac{dp_b}{dc} < 0 \).

To conclude this section, note that with \( \alpha = 1/2 \), \( k_g = k_b \) and \( c = 0 \) in equilibrium \( \theta_g = \theta_b \) and \( \pi = 1/2 \); the two sectors are identical and there is no wage inequality because \( p_g = p_b \). Second, if \( k_g > k_b \) and/or if \( c > 0 \), \( \pi > 1/2 \) and there is wage inequality resulting from the fact that \( p_g > p_b \). Given rent sharing, wage inequality arises entirely as a result of the difference in investment costs between the sectors (to either workers or firms) and their subsequent effect on prices via the product market. In a way the model provides a net supply of skills argument to wage inequality. The more it costs to produce in a sector, the lower the supply of goods in that sector, and given the demand side the higher the prices and wages. Finally, note that if prices were exogenous it would not be possible to guarantee an equilibrium with an interior solution for \( \pi \). For example, if \( \rho = 1 \) (the intermediate goods are perfect substitutes) \( p_b = \alpha \) and \( p_g = 1 - \alpha \). In this case with no price effects \( \pi \) would be undetermined.

### 4 Efficiency

This section compares the decentralized equilibrium with an allocation that maximizes net total output. It is assumed that a social planner determines time paths of values \((\theta_b, \theta_g, \pi)\) that maximize the present value of the net total surplus of the economy subject to the same restrictions that govern private decisions, that is search frictions and the existence of a product market. The planner’s problem without considering the time dependence of variables is

\[
\max \int_0^\infty \left\{ (\pi - u_b) (p_b - r k_b) + (1 - \pi - u_g) (p_g - r k_g) - \theta_b u_b r k_b - \theta_g u_g r k_g - (1 - \pi) rc \right\} e^{-rt} dt
\]

subject to

\[
\dot{u}_b = s(\pi - u_b) - q(\theta_b) \theta_b u_b \\
\dot{u}_g = s(1 - \pi - u_g) - q(\theta_g) \theta_g u_g
\]

The flow of net output of the economy at any point in time consists of the number of workers in good-jobs \((1 - \pi - u_g)\) times their net output \((p_g - r k_g)\), plus the number of workers in bad-jobs \((\pi - u_b)\) times their net output \((p_b - r k_b)\), minus the flow costs of vacancy creation \((\theta_i u_i r k_i)\), minus the flow investment cost for workers in the \( g \) sector \((1 - \pi) rc \).
The current-value Hamiltonian with multipliers $\lambda_b$ and $\lambda_g$ can be expressed as

$$H_c \equiv H_v = \left( \pi - u_b \right) \left( p_b - r k_b \right) + \left( 1 - \pi - u_g \right) \left( p_g - r k_g \right) - u_b \theta_b r k_b - u_g \theta_g r k_g - (1 - \pi) r c + \lambda_b \left( s(\pi - u_b) - q(\theta_b) \theta_b u_b \right) + \lambda_g \left( s(1 - \pi - u_g) - q(\theta_g) \theta_g u_g \right).$$

In what follows there are presented the necessary conditions that solve the optimum control planner’s problem. First consider the steady state costate equation conditions,

$$\frac{\partial H_c}{\partial u_b} = r \lambda_b = - \left( p_b - r k_b \right) - \theta_b r k_b - \lambda_b \left( s + q(\theta_b) \theta_b \right) \tag{22}$$

$$\frac{\partial H_c}{\partial u_g} = r \lambda_g = - \left( p_g - r k_g \right) - \theta_g r k_g - \lambda_g \left( s + q(\theta_g) \theta_g \right). \tag{23}$$

The necessary conditions for $\theta_b$, $\theta_g$ and $\pi$ are,

$$\frac{\partial H_c}{\partial \theta_b} = -r k_b - q(\theta_b) \left( 1 - \eta(\theta_b) \right) \lambda_b = 0 \tag{24}$$

$$\frac{\partial H_c}{\partial \theta_g} = -r k_g - q(\theta_g) \left( 1 - \eta(\theta_g) \right) \lambda_g = 0 \tag{25}$$

$$\frac{\partial H_c}{\partial \pi} = (p_b - r k_b) - (p_g - r k_g) + r c + s \left( \lambda_b - \lambda_g \right) = 0. \tag{26}$$

where $\eta(\theta_i) = -\frac{d(\theta_i)}{d(\theta_i)} \theta_i < 0$ is the elasticity of the matching function for vacancies. The optimality conditions for labor market tightness indicate that the number of jobs in each sector has to be determined by equating vacancy creation costs to the average expected value of the matches taking into account congestion externalities (which reduce matching rates by $q(\theta_i) \eta(\theta_i)$).

According to equation (26) the planner sets the skill-composition taking into account the investment costs for workers and the difference in the social value of output associated to the workers’ investment decision. A higher fraction of educated workers means an increase in the net surplus of the match $\left( \left( p_g - r k_g \right) - \left( p_b - r k_b \right) \right)$ if employed but an increase in the cost of being unemployed if the match is destroyed $\left( - s \left( \lambda_b - \lambda_g \right) \right)$.

The next step is to evaluate the social allocation equations in the decentralized equilibrium. The Hosios condition is imposed in the two sectors, that is $\beta = \eta(\theta_i)$. This makes it possible to

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9It is very difficult to obtain analytical solutions for the sufficient conditions for a maximization in this problem as it is done in Acemoglu and Shimer (1999) and Blázquez and Jansen (2003). However, for this case focusing on the necessary conditions is enough to compare the social allocation with the decentralized equilibrium.

10It is obtained that price effects cancel out in all the necessary conditions. In fact, given the CRS final output technology, (2), (3), (4) and (5) it is shown in Appendix D that

$$\frac{\partial p_b}{\partial \lambda} (\pi - u_b) = - \frac{\partial p_g}{\partial \lambda} (1 - \pi - u_g)$$

where $x = u_b, u_g, \pi$. 

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identify the model inefficiencies not related to the standard congestion externalities. Then, the necessary conditions can be rewritten as

\[ rk_b = \frac{q(\theta_b)(1 - \beta)}{D_b} p_b \]
\[ rk_g = \frac{q(\theta_g)(1 - \beta)}{D_g} p_g \]
\[ rc = q(\theta_g)\beta p_g - q(\theta_b)\beta p_b + r \left( \frac{p_g}{D_g} - \frac{p_b}{D_b} \right) \]

(27)

where

\[ D_i = r + s + (1 - \beta)q(\theta_i) + \beta \theta_i q(\theta_i). \]

It turns out that the first two order conditions for efficiency are satisfied in the decentralized economy if \( \beta = \eta(\theta_i) \). However, the expression for \( \frac{\partial H}{\partial \pi} \) in (26) evaluated at the decentralized equilibrium with \( \beta = \eta(\theta_i) \), like in the last equation of (27), is negative. This means that, denoting \( S \) to the social allocation, \( \pi^S < \pi \). Indeed, note that the equilibrium and social arbitrage conditions differ by a term \( r \left( \frac{p_g}{D_g} - \frac{p_b}{D_b} \right) \). Intuitively, an additional worker in sector \( g \) changes total output \( Y \) by \( p_g - p_b \) and makes it easier for firms in that sector to be matched, and harder for firms in the low-productivity sector to contact low-skill workers. The change in the flow discounted value of matches associated to entry in sector \( g \) is represented by \( r \left( \frac{p_g}{D_g} - \frac{p_b}{D_b} \right) \). Workers do not consider this general equilibrium effect of their education decisions.

There are also other general equilibrium effects associated to the education externality. That is, the fact that (24) and (25) but \( \frac{\partial H}{\partial \pi} < 0 \) at the decentralized equilibrium means that the equilibrium \( \theta_b \) and \( \theta_g \) differ from their efficient values with \( \pi^S < \pi \). Given the comparative statics results for \( \pi \) described in Remark 1, a too high \( \pi \) means a too high \( p_g \) and a too low \( p_b \). This price "distortions" are in turn associated to a too high \( \theta_g \) and a too low \( \theta_b \). The following proposition summarizes these results.

**Proposition 1** If \( \beta = \eta(\theta_i) \), \( k_g > k_b \) and/or \( c > 0 \) the decentralized equilibrium exhibits a too high fraction of low-skill workers, with a too high labor market tightness in the high-productivity sector, and a too low one in the low-productivity sector. Denoting \( S \) to the social allocation

\[ \pi > \pi^S, \quad \theta_g > \theta^S_g, \quad \theta_b < \theta^S_b \]

It follows directly that the number of high-skill and low-skill unemployed is too low and too high, respectively,

\[ u_g < u^S_g, \quad u_b > u^S_b. \]
Note that in the absence of externalities in the determination of worker types, the standard efficiency result would apply and the decentralized equilibrium would be constrained efficient if $\beta = \eta(\theta_i)$. This would not be the case in a model with random matching and $k_g > k_b$ like Acemoglu (2001). In such model, it is not possible to make firms internalize congestion externalities by imposing a common Hosios’ value of $\beta$, and as a consequence wages are too high in the good-job sector and too low in the bad-job sector leading to an inefficient high fraction of bad-job vacancies.\footnote{Due to the random matching assumption present in Acemoglu (2001), any firm posting a vacancy in any sector increases labor market tightness making it harder for firms in both sectors to meet an unemployed worker. Then it happens that, for a given $\beta$, the amount of the surplus transferred to workers in Nash Bargaining in bad-job matches represents less than the value of the congestion externality implied by bad-job creation. The opposite occurs in the good-job sector, where the wage is higher than the one that makes firms internalize congestion externalities. Then, this leads to a too high fraction of bad-job vacancies in the economy.} These effects disappear with a sector-specific search assumption like in this paper because job creation by firms of any type only create congestion externalities to firms of the same type and then a sector-specific Hosios condition would give the right incentives to job creation in both sectors. This result is consistent with Acemoglu and Shimer (1999) and Davis (2001). However, even in an economy with sector-specific search this paper identifies another kind of inefficiency related to how worker types are determined. In a model with an endogenous allocation of workers to the sectors, there is an externality that implies that too few workers allocate to the high productivity sector. This inefficiency is increasing in the cost of investment for both firms and workers in the good-jobs sector. This externality does not obviously arise in the model with an exogenous skill composition of the workforce.

We can also relate this result to the efficiency analysis of wage inequality. Even if the Hosios condition does not hold as long as the elasticity of the matching function is the same across the sectors it will always be the case that the price difference between the sectors is higher than it should be and then there is too much wage inequality. This too unequal wage distribution does not maximize social output. A very simple system of taxes and subsidies (which implies a net subsidy to education) would restore constrained efficiency in the model. It is accepted though that from an applied point of view the policy maker would need to know the properties of the matching function in both sectors for intervention purposes. The direction of the inefficiencies arising in the two sectors when the Hosios condition does not hold are well known. A contribution of this study is to present an additional force that would affect the efficiency properties of the equilibrium.
5 Conclusions

This paper presents a two-sector matching model with heterogenous firms and workers and analyzes the efficiency properties of the equilibrium under the Hosios' condition. Given that search is sector specific, search externalities are correctly internalized under the Hosios' condition. However, due to an education externality, the skill composition is inefficiently biased toward low skilled workers. This inefficiency translates via price effects into a too high labor market tightness in the high productivity sector and a too low one for the other sector in equilibrium relative to the planner’s allocation. The magnitude of these distortions is increasing in the education cost.

References


Appendix A. Existence and Uniqueness of the Equilibrium

The equilibrium values of vacancies with the equilibrium value of unemployment substituted in leads to,

\[ rJ^V_i = rk_i = \frac{q(\theta_i)(1 - \beta)}{D_i} p_i. \]

Taking derivatives,

\[
\frac{\partial rJ^V_i}{\partial \theta_i} = \frac{q(\theta_i)(1 - \beta)}{D_i^2} \left\{ p_i \left[ \frac{q'(\theta_i)}{q(\theta_i)} (r + s) - q(\theta_i)\beta \right] + \frac{\partial p_i}{\partial \theta_i} D_i \right\} < 0
\]

\[
\frac{\partial rJ^V_i}{\partial \theta_j} = \frac{q(\theta_i)(1 - \beta)}{D_i} \frac{\partial p_i}{\partial \theta_j} > 0.
\]

By the implicit function theorem

\[
\frac{d \theta_b}{d \theta_g B} = -\frac{\frac{\partial rJ^V_i}{\partial \theta_b}}{\frac{\partial rJ^V_i}{\partial \theta_g}} = -\frac{-\frac{\partial p_b}{\partial \theta_g}}{\frac{\partial rJ^V_i}{\partial \theta_b}} (q(\theta b)/(q(\theta b)) (r + s) - q(\theta b)\beta + \frac{\partial p_b}{\partial \theta_g}) > 0
\]

\[
\frac{d \theta_b}{d \theta_g G} = -\frac{\frac{\partial rJ^V_i}{\partial \theta_g}}{\frac{\partial rJ^V_i}{\partial \theta_b}} = -\frac{-\frac{\partial p_g}{\partial \theta_b}}{\frac{\partial rJ^V_i}{\partial \theta_g}} (q(\theta g)/(q(\theta g)) (r + s) - q(\theta g)\beta + \frac{\partial p_g}{\partial \theta_b}) > 0
\]

\[
\frac{r k_b}{q(\theta b)(1 - \beta)} \left[ \frac{q'(\theta b)}{q(\theta b)} (r + s) - q(\theta b)\beta \right] + \frac{\partial p b}{\partial \theta b} \}
\]

\[
\frac{\partial p b}{\partial \theta b}.
\]

Given the CRS in the final output technology and that intermediate goods are sold in perfectly competitive markets, it is always the case that \( \frac{\partial p_i}{\partial \theta_j} = -Y. \frac{\partial p j}{\partial \theta j} \) (see Appendix B below for a proof).

Then, it follows that

\[
\frac{d \theta_b}{d \theta g B} = \frac{Y_b \frac{\partial p b}{\partial \theta b}}{Y_g \frac{\partial p b}{\partial \theta b}} / \left\{ \frac{r k_b}{q(\theta b)(1 - \beta)} \left[ \frac{q'(\theta b)}{q(\theta b)} (r + s) - q(\theta b)\beta \right] + \frac{\partial p b}{\partial \theta b} \right\} < 1
\]

\[
\frac{d \theta_b}{d \theta g G} = \frac{Y_b \frac{r k_g}{q(\theta g)(1 - \beta)} \left[ \frac{q'(\theta g)}{q(\theta g)} (r + s) - q(\theta g)\beta \right] + \frac{\partial p g}{\partial \theta g} \right\} / \frac{\partial p g}{\partial \theta g} > 1,
\]

which imply that

\[
\frac{d \theta b}{d \theta g B} - \frac{d \theta b}{d \theta g G} < 0.
\]

Then the good job locus is always steeper than the bad job locus assuring that the equilibrium is unique. Note also that for the case of perfect substitutes (\( \rho = 1 \)) the bad job locus is a horizontal
line and the god job locus is a vertical line. In this case there are no price effects and the equilibrium may not be interior.

Note that as \( \theta_b \to 0 \) production is concentrated mostly in the good job sector which has \( \theta_g > 0 \) and finite. In that case \( p_b \to 0, u_b \to \pi \). Then firms will not open vacancies in sector \( b \) and only the good job sector prevails. In that situation \( p_g = (1 - \alpha)^{1/\rho} \) and the equilibrium value of vacancies is obviously \( rJ_g = r_k_g = \frac{q(\theta_g)(1-\beta)(1-\alpha)^{1/\rho}}{r + \alpha(1-\beta)q(\theta_g) + \beta q(\theta_g)} \). The opposite occurs when \( \theta_g \to 0 \).

Then, for any given \( \pi \) the good job locus is always steeper than the bad job locus assuring that the equilibrium is unique. It is concluded that for every \( \pi \) there exists a unique equilibrium pair \((\theta_g, \theta_b)\). It is left to show that there is a \( \pi \) such that the triplet \((\theta_g, \theta_b, \pi)\) satisfies not only the free entry conditions but also (21). To prove this, consider that \( \frac{\partial rJ_b}{\partial \pi} > 0 \) and \( \frac{\partial rJ_g}{\partial \pi} < 0 \) which in turn implies that both the good- and bad-job loci shift to the right with increases in \( \pi \). Indeed, in a version of the model with \( \pi \) given, \( \frac{\partial \theta_g}{\partial \pi} > 0 \) and \( \frac{\partial \theta_b}{\partial \pi} < 0 \). This means that for higher \( \pi \), the equilibrium pair \((\theta_g, \theta_b)\) involves higher values of \( \theta_g \) and lower values of \( \theta_b \). Then we have that \( rJ_g - rJ_b \) is monotonically increasing in \( \pi \). Finally, given that \( rJ_g - rJ_b \) is negative for \( \pi \to 0 \) and positive for \( \pi \to 1 \) there is one and only one \( \pi \) that satisfies the workers’ indifference condition.

**Appendix B. Price Effects in the Equilibrium Analysis**

I first present a proof of \( \frac{\partial p_g}{\partial \pi} = -\frac{Y_b}{Y_g} \frac{\partial \theta_g}{\partial \pi} \). Using \( \eta (\theta_i) = -\frac{q(\theta_i)}{q(\theta_g)} \theta_i < 0 \) and recalling that in equilibrium \( Y_b = \frac{q(\theta_b)\theta_g \pi}{q(\theta_g)\theta_b + \sigma} \) and \( Y_g = \frac{q(\theta_b)\theta_g (1-\pi)}{q(\theta_g)\theta_g + \sigma} \), the following results are also useful:

\[
\frac{\partial Y_b}{\partial \theta_b} = Y_b \left( \frac{1 - \eta (\theta_b)}{(q(\theta_g)\theta_b + \sigma) \theta_b} \right) \\
\frac{\partial Y_g}{\partial \theta_g} = Y_g \left( \frac{1 - \eta (\theta_g)}{(q(\theta_g)\theta_g + \sigma) \theta_g} \right) \\
\frac{\partial Y}{\partial \theta_b} = p_b \frac{\partial Y_b}{\partial \theta_b} + \frac{\partial Y_g}{\partial \theta_g} = p_g \frac{\partial Y_g}{\partial \theta_g} 
\]

\[12\] Indeed, following an analysis similar to the one of Appendix C below to sign the expressions, it is easy to show that

\[
\frac{d \theta_g}{d \pi} = \frac{\partial rJ^V}{\partial \theta_g} \frac{\partial rJ^V}{\partial \pi} - \frac{\partial rJ^V}{\partial \theta_g} \frac{\partial rJ^V}{\partial \pi} > 0 
\]

and

\[
\frac{d \theta_b}{d \pi} = \frac{\partial rJ^V}{\partial \theta_g} \frac{\partial rJ^V}{\partial \pi} - \frac{\partial rJ^V}{\partial \theta_g} \frac{\partial rJ^V}{\partial \pi} < 0. 
\]
Then, from (16) we have

\[
\begin{align*}
\frac{\partial p_g}{\partial \theta_g} &= (1 - \alpha)(\rho - 1)Y_2^{\rho - 2}Y^{1-\rho} \frac{\partial Y_2}{\partial \theta_g} + (1 - \alpha)Y_2^{\rho - 1}(1 - \rho)Y^{-\rho} \frac{\partial Y}{\partial \theta_g} \\
&= (\rho - 1)p_g s \frac{1 - \eta(\theta_g)}{(q(\theta_g)\theta_g + s)\theta_g}Y_2 \alpha \left(\frac{Y_b}{Y}\right)^\rho \\
\frac{\partial p_b}{\partial \theta_g} &= \alpha(1 - \rho)Y_2^{\rho - 1}Y^{-\rho} \frac{\partial Y}{\partial \theta_g} \\
&= (1 - \rho)p_b s \frac{1 - \eta(\theta_g)}{(q(\theta_g)\theta_g + s)\theta_b}(1 - \alpha)\left(\frac{Y_2}{Y}\right)^\rho \frac{Y_g}{Y_b},
\end{align*}
\]

which confirms that \(\frac{\partial p_g}{\partial \theta_g} = \frac{\partial p_b}{\partial \theta_g} Y_2/Y_b\) as claimed in the paper. In a similar way,

\[
\begin{align*}
\frac{\partial p_b}{\partial \theta_b} &= (\rho - 1)p_b s \frac{1 - \eta(\theta_b)}{(q(\theta_b)\theta_b + s)\theta_b}(1 - \alpha)\left(\frac{Y_2}{Y}\right)^\rho \\
\frac{\partial p_g}{\partial \theta_b} &= (1 - \rho)p_g s \frac{1 - \eta(\theta_b)}{(q(\theta_b)\theta_b + s)\theta_b}(1 - \alpha)\left(\frac{Y_2}{Y}\right)^\rho Y_g/Y_b,
\end{align*}
\]

and also \(\frac{\partial p_b}{\partial \theta_b} = -\frac{\partial p_g}{\partial \theta_b} Y_2/Y_b\).

**Appendix C. Proof of Remark 1**

We totally differentiate equations (19), (20) and (21). Given that \(rJ_i^V = \frac{\beta}{1-\beta}rJ_i^V\), the non arbitrage condition can be conveniently rewritten as \(\theta_g = \frac{c}{k_g} \frac{(1-\beta)}{\beta} + \frac{k_b}{k_g} \theta_b\).

\[
\begin{align*}
d r_k_g &= \frac{\partial rJ_g}{\partial \theta_g} d\theta_g + \frac{\partial rJ_g}{\partial \theta_b} d\theta_b + \frac{\partial rJ_g}{\partial \pi} d\pi \\
d r_k_b &= \frac{\partial rJ_b}{\partial \theta_g} d\theta_g + \frac{\partial rJ_b}{\partial \theta_b} d\theta_b + \frac{\partial rJ_b}{\partial \pi} d\pi \\
d r_c &= \frac{\beta}{1-\beta} r_k_g d\theta_g - \frac{\beta}{1-\beta} r_k_b d\theta_b + 0 d\pi
\end{align*}
\]

Then,

\[
A \begin{bmatrix} \frac{d\theta_g}{d\pi} \\ \frac{d\theta_b}{d\pi} \\ \frac{d\theta_g}{d\pi} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} \frac{d\theta_g}{d\pi} \\ \frac{d\theta_b}{d\pi} \\ \frac{d\theta_c}{d\pi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

where

\[
A = \begin{bmatrix}
\frac{\partial rJ_g}{\partial \theta_g} & \frac{\partial rJ_g}{\partial \theta_b} & \frac{\partial rJ_g}{\partial \pi} \\
\frac{\partial rJ_b}{\partial \theta_g} & \frac{\partial rJ_b}{\partial \theta_b} & \frac{\partial rJ_b}{\partial \pi} \\
\frac{\beta}{1-\beta} r_k_g & -\frac{\beta}{1-\beta} r_k_b & 0
\end{bmatrix}.
\]
We know from Appendix A that $\frac{\partial J^V}{\partial \theta_i} < 0$ and it is easy to show that $\frac{\partial J^V}{\partial \pi} > 0$ and $\frac{\partial J^V}{\partial \pi} < 0$. Given that $\det A > 0$ (see proof below), we have

$$
\begin{align*}
\frac{d\theta_g}{drk_g} &= -\frac{\frac{\partial J^V}{\partial \pi}}{\det A} \frac{\beta r k_g}{1-\beta} < 0 \\
\frac{d\theta_b}{drk_g} &= \frac{\frac{\partial J^V}{\partial \pi}}{\det A} \frac{\beta r k_g}{1-\beta} < 0 \\
\frac{d\pi}{drk_g} &= \frac{1}{\det A} \left( -\beta \frac{\partial J^V}{\partial \theta_g} r k_b + \frac{\partial J^V}{\partial \theta_b} r k_g \right) > 0 \\
\frac{d\theta_g}{dr} &= \frac{\frac{\partial J^V}{\partial \theta_g}}{\det A} - \frac{\frac{\partial J^V}{\partial \theta_i}}{\det A} \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_b} > 0 \\
\frac{d\theta_b}{dr} &= \frac{\frac{\partial J^V}{\partial \theta_g}}{\det A} - \frac{\frac{\partial J^V}{\partial \theta_i}}{\det A} \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_i} < 0 \\
\frac{d\pi}{dr} &= \frac{\frac{\partial J^V}{\partial \theta_g}}{\det A} - \frac{\frac{\partial J^V}{\partial \theta_i}}{\det A} \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_i} > 0
\end{align*}
$$

To sign $\frac{d\pi}{drk_g}$ it is easy to show that $|\frac{\partial J^V}{\partial \pi}| > |\frac{\partial J^V}{\partial \theta_i}|$. To sign $\frac{d\pi}{dr}$ we use from Appendix A the derivations used to prove that $\frac{\partial J^V}{\partial \theta_i} < 1$ and $\frac{\partial J^V}{\partial \theta_i} > 1$. The signs of $\frac{d\theta_g}{dr}$ and $\frac{d\theta_b}{dr}$ depend on the sign of $\frac{\partial J^V}{\partial \theta_i} - \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_i}$ and $\frac{\partial J^V}{\partial \theta_i} - \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_i}$, which according to points 1. and 2. of the proof of $\det A > 0$ below are positive and negative, respectively.

**Proof that** $\det A > 0$

$$
\det A = rk_g \frac{\beta}{1-\beta} \left( \frac{\partial J^V}{\partial \theta_i} \frac{\partial J^V}{\partial \theta_b} - \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_b} \right) + rk_b \frac{\beta}{1-\beta} \left( \frac{\partial J^V}{\partial \theta_i} \frac{\partial J^V}{\partial \theta_b} - \frac{\partial J^V}{\partial \pi} \frac{\partial J^V}{\partial \theta_b} \right)
$$

It can be shown that both terms in parenthesis are positive. To prove this use $\frac{\partial \theta_i}{\partial \theta_j} = -\frac{\chi_i}{\chi_j} \frac{\partial \theta_i}{\partial \theta_j}$ and the expressions for $\frac{\partial J^V}{\partial \theta_i}$ and $\frac{\partial J^V}{\partial \theta_j}$ from Appendix A to obtain the sign of the following objects:

1.

$$
\frac{\partial J^V}{\partial \theta_i} \frac{\partial J^V}{\partial \theta_g} - \frac{\partial J^V}{\partial \theta_i} \frac{\partial J^V}{\partial \theta_b} = -\frac{q(\theta_b)(1-\beta)}{D_g} \frac{q(\theta_b)(1-\beta)}{\partial \theta_i} \left\{ \frac{p_b \left[ \frac{q(\theta_b)}{q(\theta_b)} (r + s) - q(\theta_b) \beta \right]}{D_b^2} \right\} > 0
$$
Appendix D. Price Effects in the Efficiency Analysis

In footnote 11, I claim that \( \frac{\partial p_b}{\partial \pi} (\pi - b) = -\frac{\partial p_g}{\partial \pi} (1 - \pi - g) \) for \( x = b, g, \pi \).

Recall that \( Y_b = \pi - b \) and \( Y_g = 1 - \pi - g \). Using \( \frac{\partial Y_b}{\partial \pi} = 1, \frac{\partial Y_g}{\partial \pi} = -1, \frac{\partial Y}{\partial \pi} = p_b - p_g \) we have

\[
\frac{\partial p_b}{\partial \pi} Y_b = (\rho - 1) p_b \frac{\partial Y_b}{\partial \pi} + p_b Y_b (1 - \rho) \frac{\partial Y}{\partial \pi} = (\rho - 1) \left[ p_b (1 - \alpha) \left( \frac{Y_g}{Y} \right)^{\rho} + p_g \alpha \left( \frac{Y_b}{Y} \right)^{\rho} \right]
\]

and

\[
\frac{\partial p_g}{\partial \pi} Y_g = (\rho - 1) p_g \frac{\partial Y_g}{\partial \pi} + p_g Y_g (1 - \rho) \frac{\partial Y}{\partial \pi} = (\rho - 1) \left[ p_g \alpha \left( \frac{Y_b}{Y} \right)^{\rho} + (1 - \alpha) \left( \frac{Y_g}{Y} \right)^{\rho} p_b \right]
\]

Similar results can be obtained for \( x = b, g \) noting that \( \frac{\partial Y_i}{\partial u_i} = -1, \frac{\partial Y_i}{\partial u_i} = 0, \frac{\partial Y}{\partial u_i} = -p_i \). For instance,

\[
\frac{\partial p_b}{\partial u_b} Y_b = (\rho - 1) p_b \frac{\partial Y_b}{\partial u_b} + p_b Y_b (1 - \rho) \frac{\partial Y}{\partial u_b} = (1 - \rho) p_b (1 - \alpha) \left( \frac{Y_g}{Y} \right)^{\rho}
\]

\[
\frac{\partial p_g}{\partial u_b} Y_g = p_g Y_g (1 - \rho) \frac{\partial Y}{\partial u_b} = (\rho - 1) p_b (1 - \alpha) \left( \frac{Y_g}{Y} \right)^{\rho}.
\]